Theoretical and Mathematical Physics, **195**(3): 923–937 (2018)

INVERSE PROBLEM FOR A SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS FOR SH WAVES IN A VISCO-ELASTIC POROUS MEDIUM: GLOBAL SOLVABILITY

D. K. Durdiev* **and A. A. Rahmonov***

We consider a system of hyperbolic integro-differential equations for SH waves in a visco-elastic porous medium. The inverse problem is to recover a kernel (memory) in the integral term of this system. We reduce this problem to solving a system of integral equations for the unknown functions. We apply the principle of contraction mappings to this system in the space of continuous functions with a weight norm. We prove the global unique solvability of the inverse problem and obtain a stability estimate of a solution of the inverse problem.

Keywords: integro-differential equation, inverse problem, Dirac delta function, kernel, hyperbolic equation, Lame coefficient, global solvability, weight function

DOI: 10.1134/S0040577918060090

1. Setting up the problem and main results

We consider a system of integro-differential hyperbolic equations of the type

$$\rho_{\mathrm{s}}\frac{\partial^2 U}{\partial t^2}=\frac{\partial}{\partial z}\Sigma(z,t)-\chi\rho_\ell^2\biggl(\frac{\partial U}{\partial t}-\frac{\partial V}{\partial t}\biggr), \tag{1.1}$$

$$\frac{\partial V}{\partial t}=\chi\rho_\ell(U-V),\quad x,z\in\mathbb{R},\quad x>0,\quad z>0, \tag{1.2}$$

with the initial and boundary conditions

$$U|_{t\le 0}=\frac{\partial U}{\partial t}\bigg|_{t\le 0}\equiv 0,\qquad V|_{t\le 0}\equiv 0,\qquad \Sigma(+0,t)=\delta'(t). \tag{1.3}$$

Here, $U(z,t)$ is the velocity of the elastic porous body with a constant partial density ρ_{s}, $V(z,t)$ is the velocity of the fluid with a constant density ρ_ℓ, χ is coefficient of intercomponent friction (assumed to be constant and positive everywhere in this paper), and $\delta'(t)$ is the derivative of the Dirac delta function. The tension $\Sigma(z,t)$ is related to $U(z,t)$ by the formula

$$\Sigma(z,t)=\mu(z)\frac{\partial U}{\partial z}+\int_0^t k(t-\tau)\mu(z)\frac{\partial U}{\partial z}(z,\tau)\,d\tau, \tag{1.4}$$

where $\mu(z)>0$ is the Lame coefficient and $k(t)$ is the function characterizing viscosity of the medium.

*Bukhara State University, Bukhara, Uzbekistan, e-mail: durdiev65@mail.ru, araxmonov@mail.ru.

Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 195, No. 3, pp. 491–506, June, 2018. Original article submitted October 6, 2017.

The problem of determining U and V from (1.1)–(1.4) with given ρ_s, ρ_ℓ, χ, $\mu(z)$, and $k(t)$ is called the direct problem for a visco-elastic porous medium. We formulate the inverse problem: to determine the kernel $k(t)$, $t > 0$, in (1.1) via (1.4) if for the solution of the direct problem, we know that

$$U|_{z=0} = f(t), \tag{1.5}$$

where $f(t)$ is a given function.

A porous medium consisting of a visco-elastic deformable substance filled with a liquid is a more realistic model and allows explaining the observed properties of rocks in seismic investigations. Recently, mathematical modeling of wave propagation in a liquid in a saturated porous medium has received considerable attention because it has practical application in problems of geophysics, biomechanics, oil production, etc. Problems in which porosity must be taken into account, liquid saturation of the medium arises, in particular, in exploration geophysics in the search for oil layers and in the choice of parameters of wave influence on oil and gas fields to identify possible production volume.

Equations (1.1), (1.2), and (1.4) describe the propagation of elastic SH (transverse) waves in a porous medium with memory in the one-dimensional case (in terms of spatial variables). We note that the direct and inverse dynamical problems for these equations for $k(t) \equiv 0$ were studied in [1], [2]. We also note that in the absence of porosity $\rho_\ell = 0$, $V(z,t) = 0$, inverse problem (1.1)–(1.5) becomes completely analogous to the problem studied in [3]. In [4], the inverse problems of determining the density and elastic moduli from the system of elasticity equations for isotropic and anisotropic medium were investigated.

Studying inverse problems for hyperbolic integro-differential equations is an object of investigation by many authors [5]–[9] that is closest to the present work. In [5], the direct and inverse problems for a second-order hyperbolic equation with an integral term of convolution type by the time variable and one-dimensional memory function of the medium were considered. By the Fourier method, the problem reduces to a system of integral equations of the Volterra type with respect to unknown functions depending on the time variable. In [6]–[8] (also see the references therein), problems of determining the multidimensional kernel of visco-elasticity equations were studied. In [3] and [9], problems of recovering the one-dimensional kernel of the visco-elasticity equation were respectively studied in bounded and unbounded domains, and theorems on the unique solvability of these problems in the class of continuous functions with a weight norm were proved. The main feature, appropriate to [6]–[9] and to the present work, is the use of a source localized on the boundary of the considered space domain; this source initiates the physical process of wave transmission. This feature essentially increases the meaning of the investigation for applications.

We introduce a new variable x according to the equality

$$x = \psi(z) := \int_0^z \frac{d\xi}{c_t(\xi)}, \qquad c_t(z) := \sqrt{\frac{\mu(z)}{\rho_s}}. \tag{1.6}$$

We let $\psi^{-1}(x)$ denote the function the inverse to $\psi(z)$ and set

$$\begin{aligned} &\widetilde{U}(x,t) := U(\psi^{-1}(x),t), \quad \widetilde{V}(x,t) := V(\psi^{-1}(x),t), \\ &\frac{\chi\rho_\ell^2}{\rho_s} = \lambda, \qquad \sigma(x) := \sqrt{\rho_s\mu(x)}. \end{aligned} \tag{1.7}$$

Our main result is the following theorem on the global unique solvability of the inverse problem.

Theorem 1. *Let $T > 0$ be an arbitrary fixed number and the function $f(t)$ have the form* (*here and hereafter, $\sigma(0)$, etc., are understood as the limits as $x \to +0$*)

$$f(t) = \alpha(0)\delta(t) + \theta(t)f_0(t), \qquad \alpha(0) = -\frac{1}{\sigma(0)}, \tag{1.8}$$